Lecture 5: Partial order relation

Partial order relation: A relation R on a set X is partial order if R is reflexive, antisymmetric, and transitive. The set X with such R is called a partially ordered set or POSET.

Examples:

1. The relation ($\leq$) "less than or equal to" is partial order on $\mathbb{R}$.
2. The inclusion relation of sets ($\subseteq$) is partial order on a collection of sets.
3. The relation divisibility ($|$) is a partial order on $\mathbb{N}$.
4. The ralation divisibility ($|$) is not a partial order on $\mathbb{Z}$. As, $2|-2$ and $-2|2$ but $2 \neq -2$.

Hasse diagram:

A Hasse diagram is a graphical representation of the relation of elements of a POSET with an implied upward orientation. A point is drawn for each element of the POSET and two points x and y will be joined by a line segment with x appearing lower in the representation than the point corresponding to y if x is related to y.

Examples:

(a) Hasse Diagram for $(\{1,2,3,4\}, \leq)$.

(b) Hasse diagram representing the partial ordering $\{(a,b)|a \text{ divides } b\}$ on $\{1,2,3,4,6,8,12\}$.

(c) Hasse Diagram of $(P(\{a,b\}), \subseteq)$.

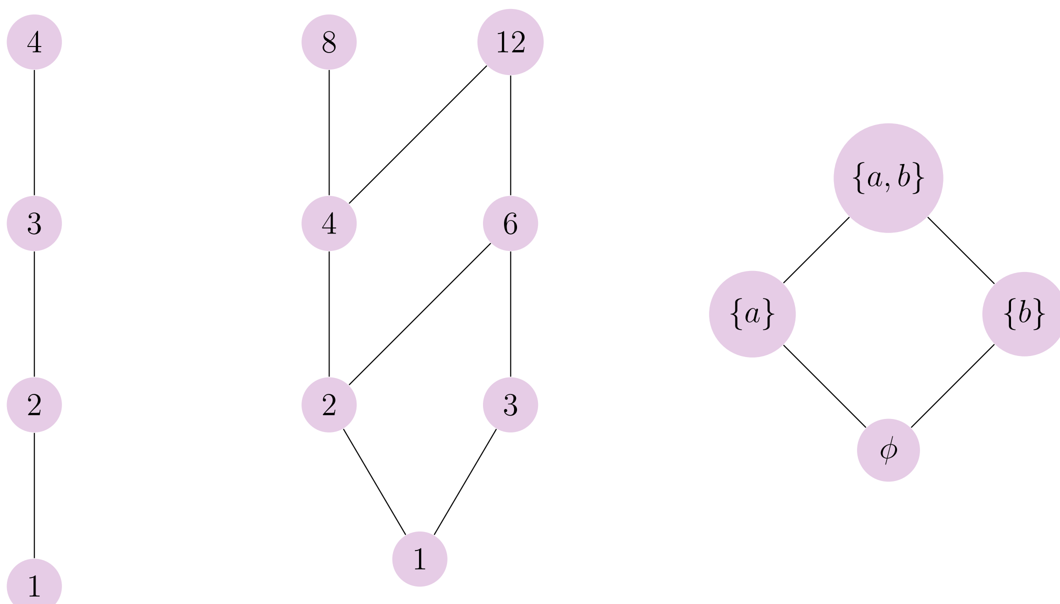

Figure 1: Hasse Diagram of (a), (b), and (c).

Totally ordered set: A POSET X with relation R is called totally ordered or chain if given every pair of $x, y \in X$, either xRy or yRx.

Examples:

1. $\mathbb{R}$ is a chain with the relation $\leq$. What if we replace $\leq$ by $<$?
2. A collection of sets with the inclusion relation $\subseteq$ is not a chain.
3. The set $S = \{2, 5, 6, 8, 9, 10\}$ is not a chain with the divisibility relation.

First and Last element: Let X be a POSET with relation R. An element $a \in X$ is the first element of X if aRx for every $x \in X$ and $b \in X$ is the last element of X if xRb for every $x \in X$.

Well ordered set: A POSET X is well ordered if every non-empty subset of X has the first element.

Example. $\mathbb{N}$ is well ordered set under the usual relation $\leq$.

Maximal and Minimal element: Let X be a POSET with relation R. An element $a \in X$ is called a minimal element of X if no element of X related to a, that is, if xRa implies $x = a$.

An element $b \in X$ is called a maximal element of X if b is not related to any element of X, that is, if bRx implies $x = b$.

Example: Consider the divisibility relation on the set $S = \{2, 3, 4, 6, 9, 10, 12, 36\}$. Then 2 and 3 are minimal elements and 10 and 36 are maximal elements.

Upper and lower bounds: Let A be a subset of a POSET X. An element $a \in X$ is called a lower bound of A if $a\,R\,x$ for every $x \in A$. An element $b \in X$ is an upper bound of A if $x\,R\,b$ for every $x \in A$. A set A may have no upper bound or lower bound.

Infimum and Supremum: Let A^* denote the collection of all upper bounds of A and A_* denote the collection of all lower bounds of A. Then the first element of A^*, if it exists, is called the least upper bound or the supremum of A. Similarly the last element of A_*, if it exists, is called the greatest lower bound or infimum of A.

Example: Let $\mathbb{R}$ with usual order relation $\leq$ and let $A = \{x \in \mathbb{R} : 1 < x < 2\}$. Here $A^* = \{x \in \mathbb{R} : x \geq 2\}$. and $A_* = \{x \in \mathbb{R} : x \leq 1\}$. So, the supremum is 2 and infimum is 1.

Order Completeness Axiom: A POSET X is said to be order complete if every non-empty subset of X which has an upper bound (or which has a lower bound) has a supremum (or infimum).

Example: The sets $\mathbb{N}$ and $\mathbb{R}$ with usual oreder $\leq$ are order complete. The set $\mathbb{Q}$ is not order complete. Consider $A := \{x \in \mathbb{Q} : 2 < x^2 < 5\}$, has no supremum and infimum.

Lattice: A POSET X is called a lattice if for every pair $x, y \in X$ has both supremum and infimum.

Example: Consider the POSETs $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\})$. Since 2 and 3 have no upper bounds in $(\{1, 2, 3, 4, 5\}, |)$, it is not a lattice. While it is easy to see that the second poset if lattice.